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Limit Surfaces for Fibrous Composite Plates

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The limit theorems of perfect plasticity and their recent extensions are used to derive limit surfaces for representative structural elements of fibrous composite plates, which are composed of long, elastic, perfectly plastic fibers site bonded or imbedded in a matrix of negligible strength. The limit surfaces are presented in the form of equations in six-dimensional generalized stress space and are completely analogous to yield surfaces for perfectly plastic structures and materials. The effects of fiber orientation geometry, fiber yielding, fiber buckling, and pullout of the fibers from the remaining structure are included in the theory and illustrated with an example.

Introduction

In a recent paper,[1] the basic theory and technique of computing limit surfaces for long-fiber composite materials was presented. It was shown that membranes composed of long, elastic, perfectly-plastic fibers, which were site bonded or imbedded in a matrix of negligible strength, could be treated by limit analysis techniques[2,3] to obtain limit surfaces for all combinations of in-plane loading. These limit surfaces are completely analogous to yield surfaces for perfectly plastic materials, and include the effect of fiber buckling and pullout of fibers from the remaining structure in addition to perfectly plastic flow.[2]

The basic tools used in Ref. 1 were the limit theorems[3] that have recently been extended to encompass phenomena other than perfect plasticity.[2] For composites, fiber buckling and pullout are admissible limit phenomena as long as they occur under constant axial load (Fig. 1). Furthermore, structural collapse that occurs as a result of any combination of acceptable limit phenomena can be handled within the framework of limit analysis. The extended limit theorems are used herein to develop equations for limit surfaces for representative structural elements of fibrous composite plates having arbitrary fiber orientation and density, and which exhaust their load-carrying capacity due to one or any combination of perfectly plastic flow of fibers, elastic buckling of fibers,[4,5] and pullout of fibers from the remaining structure. Not only are these surfaces exact for fibrous plates that are site bonded or impregnated with a matrix having negligible strength, but they are also a rigorous lower bound to the limit surfaces for fibrous plates having a matrix whose strength is not negligible. The results are, therefore, applicable to both high- and low-

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strength plastic fiber composite systems (such as steel filament-wound structures, polymer fiber reinforced epoxies, sintered metal wire plates used in structural, filtration, and noise reduction applications), and elastic-brittle fiber composite systems (glass reinforced epoxies, high-strength fiber reinforced metals) as long as these latter systems are limited by buckling or pullout before fiber fracture occurs.

It should be mentioned that other authors have used limit analysis techniques to compute strengths of composites. For example, Drucker[6] has computed the strength of rigid particle suspension in a perfectly plastic matrix under uniaxial tension. Also, Hashin,[7] Shu and Rosen,[8] and Dow et al.[9] determined the strength of uniaxially fiber reinforced composites in axial and transverse tension and shear. No attempt is made by any of the aforementioned authors, however, at deriving complete yield surfaces for structural elements of fibrous composites from a direct knowledge of the constituent material properties.

In addition, failure surfaces have been proposed for anisotropic materials by a number of researchers (Refs. 10–14), which can be applied to composite materials. Reference 10 is concerned with brittle fracture, and hence inapplicable to the limit behavior covered herein. The remaining theories can be used to empirically represent yield or limit surfaces for composites, but none can predict composite behavior knowing only the mechanical properties of constituent materials and their geometric configuration in the structure.

Since under general loading conditions plates can be subjected to membrane as well as bending stresses, limit surfaces will be developed in a six-dimensional generalized stress space described by two normal membrane forces, a membrane shear, two bending moments, and one twisting moment. An example is given to illustrate the theory.

Geometry and Force Equilibrium of a Representative Structural Element

Description of a representative structural element (RSE) of plate will follow closely the usually accepted procedure. All quantities will be related to the middle surface of the element, defined in the usual manner to be halfway between the upper and lower boundaries (Fig. 2a). Axes x and y are an orthogonal set in the plane of the middle surface. Distance normal to the center of the middle surface is given by z, measured positive as shown. An orthogonal set of unit vectors along x, y, and z are **i**, **j**, and **k**. The middle surface of the element has unit dimensions, and the total thickness of the plate is t.

All fibers in the element are essentially parallel to the middle surface, but may be arbitrarily positioned at a distance z and oriented at an angle θ measured positive counter-

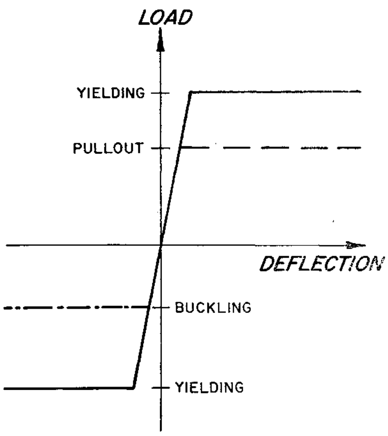

Fig. 1 Load-deflection curve for a single fiber showing ideal plasticity, fiber pullout, and fiber buckling.

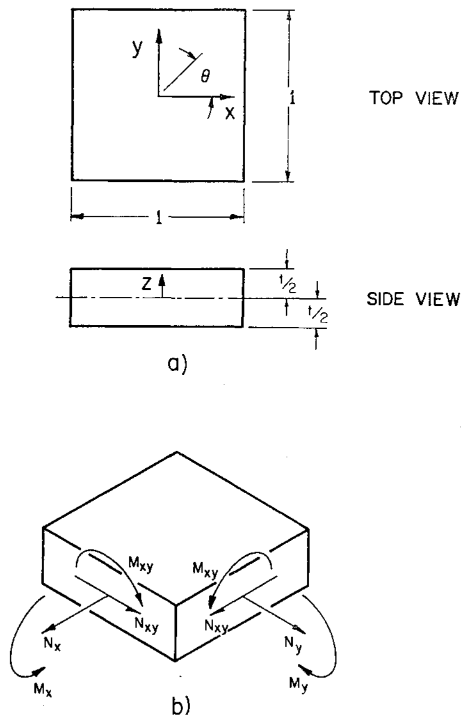

Fig. 2 a) Representative structural element of fibrous plate; b) forces acting on a plate representative structural element.

clockwise from the x axis. The fibers are elastic or elastic-perfectly plastic with behavior as illustrated in Fig. 1, and will all be assumed to have identical geometrical and material properties. However, plates containing any specific combinations of fibers having different properties can also be analyzed by the method outlined herein. Fibers are either site bonded at their crossover points or held together by a matrix material that surrounds them. If there is a matrix, its total strength is assumed to be negligible compared to that of the fibers insofar as resultant load-bearing capacity is concerned, but strong enough to transfer load to fibers and maintain fiber structural geometry. The RSE is small compared to the total structure size, but large compared to the spacing between fibers such that all typical fiber groups are contained in the element.

There will be in general a finite number of fibers in the RSE characterized by n fiber families, each ith family having a different number of fibers per unit length, ν_i', orientation θ_i and position z_i with respect to the middle surface. It was shown,[1] however, that membrane theory could be developed using a fiber density function, and that no exactness was sacrificed if this density were constructed of Dirac impulse functions. This same procedure can be generalized to the case where fiber position varies with distance from the middle surface as well as orientation angle by letting

$$\nu'(\theta,z) = \sum_{i=1}^{n} \nu_i'\delta(\theta - \theta_i, z - z_i) \qquad (1)$$

where $\nu'(\theta,z)d\theta dz$ is the number of fibers per unit length between $z - dz/2$ and $z + dz/2$ oriented at angles varying from $\theta - d\theta/2$ to $\theta + d\theta/2$, and δ is the Dirac impulse function. The development in this section, therefore, will be made using the density function $\nu'(\theta,z)$ as though it were continuous, but realizing that this covers the discrete fiber family case through Eq. (1).

The fiber length per unit area in the plane L_f, and the fiber mass per unit plate area M_f are

$$L_f = \int_{-t/2}^{t/2}\int_{-\pi/2}^{\pi/2} \nu'(\theta,z)d\theta dz \qquad (2)$$

and

$$M_f = \rho_l \int_{-t/2}^{t/2} \int_{-\pi/2}^{\pi/2} \nu'(\theta,z) d\theta dz \qquad (3)$$

The relation between the three-dimensional density function, $\nu'(\theta,z)$ for the plate, and the two-dimensional density function $\nu(\theta)$ for a membrane[1] is

$$\nu(\theta) = \int_{-t/2}^{t/2} \nu'(\theta,z) dz \qquad (4)$$

We will be considering RSE's that are undergoing collapse, and hence all or nearly all of the fibers in the RSE will be at their collapse loads. If the fiber spacing for a given family is not uniform in the RSE and fibers have the same load, there will be, in addition to force resultants, an arbitrary number of in-plane force moments of various order. These moments are analogous to multipolar stresses for continua, the simplest of which are couple stresses. Since a discussion of couple or multipolar effects is beyond the scope of this paper, we will assume that all fibers at a given angle θ and depth z are equally spaced and hence the number density ν' is not a function of x or y. Under these conditions, the most general resultants per unit length are normal forces N_x and N_y, a shear force N_{xy}, bending moments M_x and M_y, and a twisting moment M_{xy} (Fig. 2b).

Let f be the axial force in a fiber. It will be shown later that the fiber force depends only upon the fiber's depth position z and orientation θ, and therefore $f = f(\theta,z)$. We can then write the force and moment resultants per unit length on a representative structural element of plate as

$$N_x = \int_{-t/2}^{t/2} \int_{-\pi/2}^{\pi/2} f(\theta,z)\nu'(\theta,z) \cos^2\theta d\theta dz \qquad (5a)$$

$$N_y = \int_{-t/2}^{t/2} \int_{-\pi/2}^{\pi/2} f(\theta,z)\nu'(\theta,z) \sin^2\theta d\theta dz \qquad (5b)$$

$$N_{xy} = \int_{-t/2}^{t/2} \int_{-\pi/2}^{\pi/2} f(\theta,z)\nu'(\theta,z) \sin\theta \cos\theta d\theta dz \qquad (5c)$$

$$M_x = \int_{-t/2}^{t/2} \int_{-\pi/2}^{\pi/2} zf(\theta,z)\nu'(\theta,z) \cos^2\theta d\theta dz \qquad (5d)$$

$$M_y = \int_{-t/2}^{t/2} \int_{-\pi/2}^{\pi/2} zf(\theta,z)\nu'(\theta,z) \sin^2\theta d\theta dz \qquad (5e)$$

$$M_{xy} = \int_{-t/2}^{t/2} \int_{-\pi/2}^{\pi/2} zf(\theta,z)\nu'(\theta,z) \sin\theta \cos\theta d\theta dz \qquad (5f)$$

General Limit Surface

Similar to the procedure followed in Ref. 1, the limit surface for a representative structural element of plate with arbitrary fiber density and orientation will be computed using the limit theorems.[2,3] Possible mechanisms of RSE collapse will include fiber failure caused by perfectly plastic flow, constant load buckling, or constant load fiber pullout. Each fiber's limit load in tension is f^0, and in compression, $\varphi_c f^0$. Matrix material, if present, has negligible strength compared to that contributed by the fibers.

An upper bound to the limit surface is obtained by assuming an arbitrary linear velocity field for the plate RSE boundary given by

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{i} + v_z\mathbf{k} \qquad (6a)$$

$$v_x = (\dot\epsilon_x + \dot\varkappa_x z)x + (\dot\epsilon_{xy} + \dot\varkappa_{xy} z)y \qquad (6b)$$

$$v_y = (\dot\epsilon_{xy} + \dot\varkappa_{xy} z)x + (\dot\epsilon_y + \dot\varkappa_y z)y \qquad (6c)$$

$$v_z = v_z^0(x,y) \qquad (6d)$$

where v_x, v_y, and v_z are velocity components along the x, y, and z coordinate axes, and $\dot\epsilon_x$, $\dot\epsilon_y$, $\dot\epsilon_{xy}$, $\dot\varkappa_x$, $\dot\varkappa_y$, $\dot\varkappa_{xy}$ are constants representing the macroscopic middle surface extensional and shear strain rates, and the curvature and twist rates, respectively. Note that v_z is not a function of z, and that the deformation of the interior of the RSE remains unspecified. For this RSE boundary velocity, the relative deflection of fiber ends per unit length is given by

$$\dot e_a = (\dot\epsilon_x + \dot\varkappa_x z) \cos^2\theta + (\dot\epsilon_y + \dot\varkappa_y z) \sin^2\theta + 2(\dot\epsilon_{xy} + \dot\varkappa_{xy} z) \sin\theta \cos\theta \qquad (7)$$

It should be noted that if fibers are either buckling or pulling out of the structure, $\dot\epsilon_x$, $\dot\epsilon_y$, $\dot\epsilon_{xy}$, $\dot\varkappa_x$, $\dot\varkappa_y$, $\dot\varkappa_{xy}$, and $\dot e_a$ are not strictly strain rates. In these two cases, however, $\dot\epsilon_x$, $\dot\epsilon_y$, $\dot\epsilon_{xy}$, $\dot\varkappa_x$, $\dot\varkappa_y$, $\dot\varkappa_{xy}$ do determine the relative velocities of the RSE edges in the x and y directions, [Eqs. (6b) and (6c)] and as such are valid deformation measures in the limit analysis sense.[2] Similarly, even though $\dot e_a$ is not truly an axial strain rate, the product $\dot e_a l$ (l = fiber length) gives the relative deflection of fiber ends and hence is a valid deformation measure.

Since fiber extensional strain rate depends only on the macroscopic generalized strain, the depth position z, and the orientation θ of the fiber, all fibers with coordinates θ, z have the same extensional strain rate. Furthermore, since at collapse the deformation measures are those of the fiber limit state, we have

$$\dot e_a > 0, f(\theta,z) = f^0$$

(fibers are deforming plastically or pulling out in tension; fiber load is their limit load in tension)

$$\dot e_a < 0, f(\theta,z) = -\varphi_c f^0$$

(fibers are deforming plastically or buckling in compression; fiber load is their limit load in compression) (8)

$$\dot e_a = 0, -\varphi_c f^0 \le f_i \le f^0$$

(fibers are not deforming; fiber load may be anywhere between limit load in tension and compression)

The rate of work done by the external forces in the assumed velocity field is

$$W = N_x\dot\epsilon_x + N_y\dot\epsilon_y + 2N_{xy}\dot\epsilon_{xy} + M_x\dot\varkappa_x + M_y\dot\varkappa_y + 2M_{xy}\dot\varkappa_{xy} \qquad (9)$$

The rate of internal work D is

$$D = \sum_{\substack{\text{all} \\ \text{fibers}}} \underset{\substack{\text{force} \\ \text{in fiber}}}{f} \underset{\substack{\text{relative motion} \\ \text{of fiber ends}}}{\Delta}$$

The relative motion of fiber ends may be written $\Delta = \dot e_a l$, even in the case where fibers are buckling or pulling out, and hence

$$D = \int_{-t/2}^{t/2} \int_{-\pi/2}^{\pi/2} f(\theta,z)\dot e_a(\theta,z)\nu'(\theta,z) d\theta dz \qquad (10)$$

where $\dot e_a$ is given by Eq. (7) and f is given by Eq. (8). Equating D to W, and noting that since $\dot\epsilon_x$, $\dot\epsilon_y$, $\dot\epsilon_{xy}$, $\dot\varkappa_x$, $\dot\varkappa_y$, $\dot\varkappa_{xy}$ are arbitrary, their coefficients must be equal, the equilibrium equations (5) are recovered. An upper bound to the true limit surface, then, is given by Eqs. (5) with fiber force determined by Eqs. (7) and (8).

To obtain a lower bound to the limit surface, we assume the fiber force field given by Eqs. (7) and (8), which, when combined with the equilibrium Eq. (6), coincides with the upper bound.

It is concluded, therefore, that Eqs. (5, 7, and 8) constitute the true limit surface for an RSE of plate having arbitrary fiber density and orientation. At any distance z from the middle surface of the plate, the fiber axial strain rate will vanish when

$$\theta = \alpha_1, \alpha_2 = \tan^{-1} \times \left\{ -\frac{(\rho_1 + \rho_4 z)}{(1 + \rho_3 z)} \pm \left[\left(\frac{\rho_1 + \rho_4 z}{1 + \rho_3 z} \right) - \frac{(\rho_2 + \rho_5 z)}{(1 + \rho_3 z)} \right]^{1/2} \right\} \qquad (11)$$

where

$$\rho_1 = \frac{\dot{\epsilon}_{xy}}{\dot{\epsilon}_y},\ \rho_2 = \frac{\dot{\epsilon}_x}{\dot{\epsilon}_y},\ \rho_3 = \frac{\dot{\varkappa}_y}{\dot{\epsilon}_y},\ \rho_4 = \frac{\dot{\varkappa}_{xy}}{\dot{\epsilon}_y},\ \rho_5 = \frac{\dot{\varkappa}_x}{\dot{\epsilon}_y} \quad (12)$$

The two angles $\alpha_1(z)$ and $\alpha_2(z)$ determine, for a particular set of strain rate ratios, the boundaries of the areas where $f = f^0$ and $f = -\varphi_c f^0$. For a particular z, $-\pi/2 \leq \alpha_1, \alpha_2 \leq \pi/2$.

Note that $f(\theta,z)$ is a function of the five strain ratios, and hence so are the equilibrium equations. The final limit surface is then a parametric representation in six-dimensional N_x, N_y, N_{xy}, M_x, M_y, M_{xy} space in terms of the five strain ratios $\rho_1, \ldots, \rho_5$.

The procedure to determine the limit surface of an RSE of plate having a known $\nu'(\theta,z)$ is as follows. First, a set of strain ratios is chosen and α_1, α_2 are computed as functions of z from Eq. (11). The sign of $\dot{e}_a$ is determined in each region of θ, z space bounded by the curves $\alpha_1(z)$, $\alpha_2(z)$. Using Eqs. (8), the function $f(\theta,z)$ is then determined and the integrations (5) performed. The result is one point on the limit surface. Repeating this procedure for a different set of strain rate ratios determines another point on the limit surface and so on.

The limit surface determined by the aforementioned procedure is exact for plates whose fibers are held together by site bonds or a matrix with negligible strength. It is important to note, however, that the surface is also a rigorous lower bound for plates whose fibers are imbedded in a perfectly plastic matrix whose strength is not negligible. This is readily seen by assuming an axial stress in the fibers, which varies with orientation in the manner of Eqs. (7) and (8), and zero stresses in the matrix. By the lower bound theorem, the resultants found from Eqs. (5, 7, and 8) are lower bounds to the limit surface for a structure imbedded in a matrix of material which has an arbitrary limit condition. It is expected that the lower bound will be closer to the true limit surface as the strength and volume of matrix becomes smaller than the strength and volume of fibers.

Hill[15] has proven that during plastic flow, the generalized strain rate vector associated with a generic point on the yield surface of a composite is normal to the yield surface at that point. Drucker[16,17] has also shown that normality of the strain rate vector to the yield surface follows immediately if materials or structural elements obey the stability postulate. Since by definition we are considering fibers which behave in a neutrally stable way, normality is a direct consequence of our assumed material and structural behavior but could also be shown to follow directly from the governing equations (1, 2).

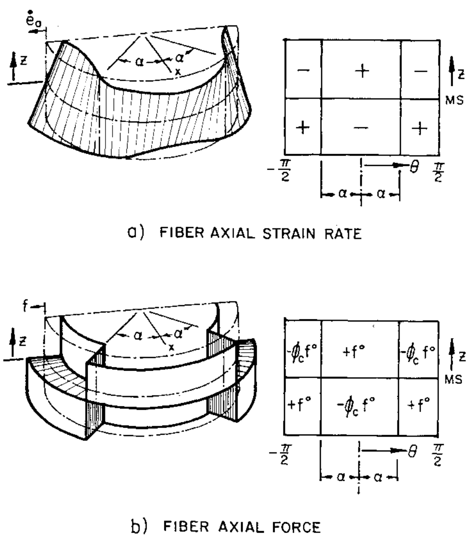

Fig. 3 Typical fiber axial strain rate and axial force for isotropic, fibrous plate in bending only.

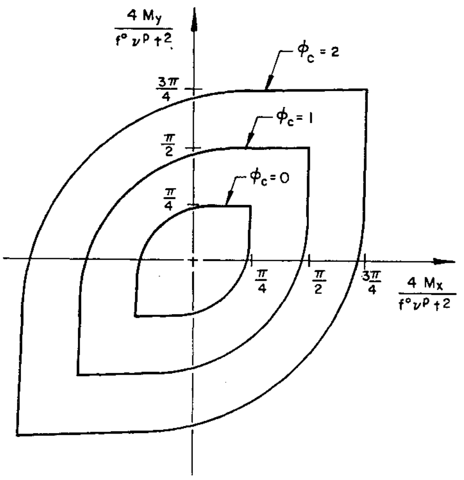

Fig. 4 Limit surface for bending of isotropic fibrous plate element.

The previously outlined procedure is very lengthy if many points on the limit surface are desired, but the above formulation lends itself well to numerical analysis on a digital computer. Many times, however, the whole limit surface is not desired, and simplifications can be made that enable a closed form solution.

Example

As an example, consider an element of fibrous plate having $\nu'(\theta,z) = \nu^p =$ constant, composing a structure in pure bending only. The plate is isotropic when ν' is constant. Suppose that restrictions on the structure (for example, axisymmetry and no in-plane loads) require that the generalized strain rates $\dot{\epsilon}_x$, $\dot{\epsilon}_y$, $\dot{\epsilon}_{xy}$, and $\dot{\varkappa}_{xy}$ vanish. The only nonzero stresses are M_x and M_y, and nonzero strain rates are $\dot{\varkappa}_x$ and $\dot{\varkappa}_y$. The equation for the $+ \; -$ crossover points α_1 and α_2 becomes

$$\alpha_1 = -\alpha_2 = \tan^{-1}\left[\left(-\frac{\dot{\varkappa}_x}{\dot{\varkappa}_y}\right)^{1/2}\right] \quad (13)$$

The fiber axial strain rate is, from Eq. (7)

$$\dot{e}_a = {}^{\cdot}{}_x z \cos^2\theta + \dot{\varkappa}_y z \sin^2\theta \quad (14)$$

The typical fiber axial strain rate is shown as a function of θ and z in Fig. 3a where the bandwidth angle, 2α, depends on $\dot{\varkappa}_x/\dot{\varkappa}_y$. The corresponding fiber axial force function is shown in Fig. 3b.

The limit surface may then be determined in terms of the angle parameter α whose range is $0 \leq \alpha \leq \pi/2$ covering all possible admissible values of $\dot{\varkappa}_x/\dot{\varkappa}_y$. Substituting the Fig. 3b axial force function into the equilibrium equation (5), and integrating, gives the limit surface in the parameter α.

Eliminating α, we obtain

$$\frac{4(M_x - M_y)}{(1 + \varphi_c)f^0\nu^p t^2} = \pm \cos\left[\frac{4(M_x + M_y)}{(1 + \varphi_c)f^0\nu^p t^2}\right] \quad (15)$$

which is shown in Fig. 4 for several values of φ_c. Values of $\varphi_c < 1$ correspond to fiber limit load being greater in tension than in compression, as in the case of tensile yielding and compressive buckling. Values of $\varphi_c > 1$ occur when fiber compressive limit load is greater than that in tension, as when fibers yield in compression but pullout in tension. It is noted that this surface has the same shape as that for an isotropic fibrous membrane[1] with zero shear ($N_{xy} = 0$) and closely resembles the parabolic limit condition proposed by von Mises.[18] It is also seen that in the case of pure bending, unequal fiber limit loads in tension and compression do not destroy yield surface symmetry as would be the case if only membrane forces were present.[1]

Conclusions

Load-bearing capacity of fibrous composite plates in the form of limit surfaces can be computed from methods and equations presented herein. Fibers that are site bonded to form a structure and those imbedded in a matrix having negligible strength can be treated as long as exhaustion of load-bearing capacity occurs for any combination of perfectly plastic flow of fibers, constant load elastic buckling of fibers, or pullout of fibers under constant load from the remaining structure. The limit mechanism and fiber limit loads must be known for the particular structure under consideration.

The complete generality of the limit surface equations in terms of the fiber orientation density makes them particularly valuable for analysis and design. A degree of complexity has been removed since the basic structural material is no longer the fibers, but the structural element having limit characteristics computed from the preceding equations. In addition, the equations make it possible for the design engineer to evaluate structural designs having different fiber orientation densities, and hence obtain optimal configurations for particular applications.

The limit surfaces derived herein also represent rigorous lower bounds to a fibrous composite plate containing a perfectly plastic matrix material whose strength is not negligible.

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